Bose-Einstein correlations
& the anomalous dimension of QCD

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• fractal structure of QCD jets – anomalous dimension
• Bose-Einstein in plane wave approximation:
  • Central limit theorem (CLT): Gaussian sources
  • Generalized CLT: Lévy stable laws
• Two and three particle correlations
• Measuring the anomalous dimension with BEC/HBT

T. Cs, S. Hegyi, W. A. Zajc, nucl-th/0402035, Nukleonika in press
T. Cs, T. Novák et al in preparation
(Multi)fractal jets in QCD

\[ dP = \frac{C \alpha_s}{2\pi} dx_1 dx_3 \frac{(x_1^2 + x_3^2)}{(1 - x_1)(1 - x_3)} \]

\[ x_2 + x_1 + x_3 = 2 \]

\[ k_\perp^2 \equiv W^2 (1 - x_1)(1 - x_3) \]

\[ y \equiv \frac{1}{2} \log \left( \frac{1 - x_1}{1 - x_3} \right) \]

\[ dP \sim \frac{C \alpha_s}{\pi} \frac{dk_\perp^2}{k_\perp^2} dy \]

\[ k_\pm \equiv k_\perp \exp(\pm y) < W \]

a triangular region in the \((y, \kappa \equiv \log(k_\perp^2))\)-plane

Figure 1: (a) The phase space available for a gluon emitted by a high energy q\bar{q} system is a triangular region in the \(y-\kappa\) plane. (b) If one gluon is emitted at \((y_1, \kappa_1)\) the phase space for a second (softer) gluon is represented by the area of this folded surface. (c) Each emitted gluon increases the phase space for the softer gluons. The total gluonic phase space can be described by this multifaceted surface.
The baseline forms a (multi)fractal, and the fractal dimension is given by the anomalous dimension of QCD: 

$$1 + (3 \frac{\alpha_s}{2 \pi})^{0.5}$$

Lund string, with string tension of 1 GeV/fm: maps the fractal in momentum space to coordinate space, without changing the fractal dimension.

Figure 1: (a) The phase space available for a gluon emitted by a high energy q\(\bar{q}\) system is a triangular region in the \(y-\kappa\) plane. (b) If one gluon is emitted at \((y_1, \kappa_1)\) the phase space for a second (softer) gluon is represented by the area of this folded surface. (c) Each emitted gluon increases the phase space for the softer gluons. The total gluonic phase space can be described by this multifaceted surface.
Selfsimilarity, Lévy stable laws

\[ x = \sum_n x_n \]

Hence the distribution of the sum \( x \) is obtained as an \( n \)-fold convolution,

\[ f(x) = \int dx_1...dx_n f_1(x_1)...f_n(x_n) \delta(x - x_1 - x_2\ldots - x_n) \]

\[ \tilde{f}(q) = \prod_{i=1}^{n} \tilde{f}_i(q) \]

\[ \tilde{f}(q) = \exp \left( iq\delta - |\gamma q|^\alpha \right), \]

\[ \tilde{f}_i(q) = \exp \left( iq\delta_i - |\gamma_i q|^\alpha \right), \]

\[ \prod_{i=1}^{n} \tilde{f}_i(q) = \exp \left( iq\delta - |\gamma q|^\alpha \right), \]

\[ \gamma^\alpha = \sum_{i=1}^{n} \gamma_i^\alpha, \]

\[ \delta = \sum_{i=1}^{n} \delta_i. \]
Bose–Einstein C: Plane wave approximation

\[ C_2(k_1, k_2) = \frac{N_2(k_1, k_2)}{N_1(k_1) N_1(k_2)} \]

**Experimental conditions:**

i) The correlation function tends to a constant for large values of the relative momentum \( q = k_1 - k_2 \).

ii) Near \( |q| = 0 \), the correlation function deviates from its asymptotic, large \(|q|\) value in a certain domain of its argument.

iii) The two-particle correlation function is related to a Fourier transformed space-time distribution of the source.

\[
S(x, k) = f(x) g(k), \quad \int dx \, f(x) = 1, \quad \int dk \, g(k) = \langle n \rangle,
\]

\[
N_1(k) = \int dx \, S(x, k) = g(k).
\]

\[
N_2(k_1, k_2) = \int dx_1 dx_2 \, S(x_1, k_1) S(x_2, k_2) |\psi_{k_1,k_2}(x_1, x_2)|^2.
\]

\[
\psi_{k_1,k_2}(x_1, x_2) = \frac{1}{\sqrt{2}} [\exp(i k_1 x_1 + i k_2 x_2) + \exp(i k_1 x_2 + i k_2 x_1)].
\]

\[
C_2(k_1, k_2) = 1 + |\tilde{f}(q)|^2,
\]

\[
\tilde{f}(q) = \int dx \, \exp(i q x) f(x), \quad q = k_1 - k_2.
\]
Extra Assumption: ANALYTICITY

\[ C_2(k_1, k_2) = 1 + |\tilde{f}(q_{12})|^2, \]

\[ \tilde{f}(q_{12}) = \int dx \exp(iq_{12}x) f(x), \quad q_{12} = k_1 - k_2. \]

\[ \tilde{f}(q) \approx 1 + iq\langle x \rangle - q^2\langle x^2 \rangle / 2 + \ldots, \]

\[ C(q) = 1 + |\tilde{f}(q)|^2 \approx 2 - q^2(\langle x^2 \rangle - \langle x \rangle^2) \approx 1 + \exp(-q^2R^2), \]

\[ R = \sqrt{\langle x^2 \rangle - \langle x \rangle^2}. \]
Limit distributions, Lévy laws

The characteristic function for limit distributions is known also in the case, when the elementary process has infinite mean or infinite variance. The simplest case, for symmetric distributions is:

\[ \tilde{f}(q) = \exp (iq\delta - |\gamma q|^\alpha) , \]

explanatory note

\[ \tilde{f}(q) \approx 1 + iqx_0 - \frac{1}{2}|qR|^\alpha \]

This is not analytic function. The only case, when it is analytic, corresponds to the \( \alpha = 2 \) case. The general form of the correlation function is

\[ C(q;\alpha) = 1 + \exp (-|qR|^\alpha) . \]

where \( 0 < \alpha \leq 2 \) is the Lévy index of stability.

4 parameters: center \( x_0 \), scale \( R \), index of stability \( \alpha \), asymmetry parameter \( \beta \)
Examples in 1d

Cauchy or Lorentzian distribution, $\alpha = 1$

$$f(x) = \frac{1}{\pi} \frac{R}{R^2 + (x - x_0)^2}, \quad -\infty < x < \infty,$$

$$C(q) = 1 + \exp \left( -|q R| \right).$$

Asymmetric Levy distribution, has a finite, one sided support, $\alpha = 1/2$, $\beta = 1$

$$f(x) = \sqrt{\frac{R}{8\pi}} \frac{1}{(x - x_0)^{3/2}} \exp \left( -\frac{R}{8(x - x_0)} \right), \quad x_0 < x$$

$$C(q) = 1 + \exp \left( -\sqrt{|q R|} \right).$$
Lévy sources and BE/HBT correlations

$\alpha = 0.4$: very peaked correlation function, strongly decreasing source density with power-law tail.

Source density (linear-linear)

Source density (log-log)

$s = \frac{r}{R}$

scaled coordinate
Lévy sources and BE/HBT correlations

\[ \alpha = 0.8: \text{ peaked correlation function, decreasing source density with power-law tail} \]
Lévy sources and BE/HBT correlations

\[ \alpha = 1.2: \text{ less peaked correlation function, less decreasing source density with power-law tail} \]
Lévy sources and BE/HBT correlations

\[ \alpha = 1.6: \text{ correlation function can be mistaken as Gaussian} \]

source by eye looks like a Gaussian on lin-lin scale, but has a power-law tail on the log-log scale

\[ s = r/R \]

classified coordinate

\[ \alpha = 1.6 \]

Source density (log-log)

Source density (linear-linear)
Lévy sources and BE/HBT correlations

\( \alpha = 2.0 \): really Gaussian correlation function, really Gaussian source density, power-law tail is gone

\( s = r/R \)

scaled coordinate

BEC/HBT

Source (log-log)

Source density
(linear-linear)
The case of symmetric Levy distributions is solved by

$$\tilde{f}(q) = \exp \left( iqx_0 - \frac{1}{2} \left| \sum_{i,j=1}^{3} R_{i,j}^2 q_i q_j \right|^{\frac{\alpha}{2}} \right)$$

with the following multidimensional Bose-Einstein correlations

$$C(q) = 1 + \exp \left( - \left| \sum_{i,j=1,3} R_{i,j}^2 q_i q_j \right|^{\frac{\alpha}{2}} \right)$$

and the corresponding space-time distribution is given by

$$f_\alpha(s(x)) = \frac{1}{(2\pi \det R^2)^{d/2}} \int_0^\infty dt \ t^{d-1} (ts(x))^{1-d/2} J_{d/2-1} (ts(x)) e^{-t^\alpha},$$

$$s(x) = |R^{-1} x| = \sqrt{\frac{r_x^2}{X^2} + \frac{r_y^2}{Y^2} + \frac{r_z^2}{Z^2}}.$$
$\alpha = 0.6$

$s = \frac{r_x^2}{X^2} + \frac{r_y^2}{Y^2}$  
scaled coordinate: 1d problem for symmetric Lévy sources
2d Lévy sources, linear scale

$\alpha = 0.8$
$\alpha = 1.2$
2d Lévy sources, linear scale

\[ \alpha = 1.6 \]
$\alpha = 2.0$

Finally, a Gaussian case!
2d Lévy sources, log-linear scale

\( \alpha = 0.6 \)

For \( \alpha < 2 \): power-law tail

\( \alpha = 2.0 \)
Asymmetric Lévy & 3-particle correlations

Normalized three-particle cumulant correlation function

\[ w(1, 2, 3) = \frac{\kappa_3(1, 2, 3)}{2\sqrt{\kappa_2(1, 2)\kappa_2(2, 3)\kappa_2(3, 1)}} \]

\[ w(1, 2, 3) = \cos \left\{ \frac{\beta}{2} R^\alpha \tan \left( \frac{\alpha \pi}{2} \right) \left[ \sum_{(i,j)} \left| q_{ij} \right|^\alpha \text{sign}(q_{ij}) \right] \right\} \]

\[ w = \cos(\phi). \]

The angle \( \phi \) is directly proportional to \( \beta \), the asymmetry
Fits to NA22 and UA1 Bose–Einstein (BEC) data

\[ \alpha_s(QCD) = \pi \alpha^2(BEC)/6 \]

UA1: \( 0.1 \sim 0.125 \pm 0.005 \)
NA22: \( 0.14 \sim 0.23 \pm 0.05 \)

within errors even the running of \( \alpha_s(QCD) \) is seen from BEC
Summary and outlook

check the existence of the Lévy exponent in collisions in p+p, d+Au and Au+Au @ RHIC,

Insert an extra parameter : Index of stability
when Gaussians start to fail
when Gaussians works seemingly well

relate $a$ to the properties of QCD
Levy index of stability $\leftrightarrow$ anomalous dimension of QCD

Prediction: running of $\alpha$(BEC) as given by the well-known $\alpha$(QCD)

Interpretation in soft Au+Au: different domain,
for far from second order phase transition, thermal source:
$\alpha=2$
but near to the critical point: related to critical exponents of the phase transition.
Really Model Independent Method

- HBT or B-E: often believed to be dull Gaussians
- In fact: may have rich and interesting structures
- Experimental conditions for model independent study:
  - $C(Q) \rightarrow \text{const for } Q \rightarrow \infty$
  - $C(Q)$ deviates from this around $Q=0$
- Expand $C(Q)$ in the abstract Hilbert space of orthogonal functions, identify the measure in $H$ with the zeroth order shape of $C(Q)$.
- Method works not only for Bose-Einstein correlations but for other observables too.
Example of NA35 S+Ag 2d correlation data, and a Gaussian fit to it (lhs) which misses the peak around $q=0$. The rhs shows a 2d Edgeworth expansion fit to E802 Si+Au data at AGS (upper panel) compared to a 2d Gaussian fit for the same data set (lower panel). Note the difference in the vertical scales.