Conservation of energy and momentum

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Newton's 2nd law: \( \vec{F} = m \vec{a} \)

Can rewrite this as \( \vec{F} = m \frac{d \vec{v}}{dt} \)

if \( m \) is constant (mass of a ball)

then \( \vec{F} = \frac{d}{dt} (m \vec{v}) \)

but this is the momentum

\( \vec{P} = \frac{d \vec{P}}{dt} \)

If the net force is zero, \( \vec{P} \)
does not change in time (it's a constant).

Newton's 3rd law:

\( \vec{F}_{A \text{ on } B} = - \vec{F}_{B \text{ on } A} \)
Apply Newtont's 3rd law to two hockey pucks that collide.

Before collision:

After collision:

We know $\vec{P}_{\text{total, i}} = \vec{P}_{\text{total, f}}$.

How do we know this?
N's 3rd: \[ \vec{F}_A \text{ on } B = -\vec{F}_B \text{ on } A \]

N's 2nd: \[ \frac{d}{dt} \vec{P}_B = - \frac{d}{dt} \vec{P}_A \]

\[ \frac{d}{dt} (\vec{P}_B + \vec{P}_A) = 0 \]

\[ \frac{d}{dt} (\vec{P}_{\text{total}}) = 0 \]

The **total** momentum does not change in time when the net force on a closed system is zero.

Momentum is **conserved** in the collision.
Two kinds of collisions

(1) **Elastic**: total momentum $\vec{P}_{\text{total}}$ and total kinetic energy ($K = \frac{1}{2}mv^2$) conserved (same before & after collision)

Example: Billiard balls

(2) **Inelastic**: only total momentum conserved.

Example: two chunks of clay that hit and stick together.

Elastic
- Billiard balls, air hockey pucks

Inelastic
- chunks of clay, car wreck
Conservation of energy

There are three kinds of energy relevant for this discussion.

1) Potential energy (gravitational)
   \[ U = mgh \]

2) Kinetic energy (from motion)
   \[ K = \frac{1}{2}mv^2 \]

3) Heat (thermal energy)
   energy lost to friction, for example.

In a closed system, the sum of all three kinds of energy is a constant (does not vary in time). No energy is lost or created.

You can convert one type of energy to another.
Bouncing ball example

ball initially at rest at height $h$.

$E_{\text{total}} = mgh = U$

(all energy is potential energy)

Let go of ball. Just before it hits the ground, the energy is all kinetic.

$E_{\text{total}} = \frac{1}{2}mv^2 = K$

Since energy is conserved, we can set this equal to $mgh$.

$E_{\text{total}, t_2} = E_{\text{total}, t_2}$

$mgh = \frac{1}{2}mv_2^2$

$v_2 = \sqrt{2gh}$
If the collision with the earth is elastic, the total kinetic energy after the collision is the same as just before it:

\[ k_{\text{before}} = k_{\text{after}} \]
\[ \frac{1}{2} m v_1^2 = \frac{1}{2} m v_2^2 \]

As the ball rises, it is being slowed down by the gravitational force. \((F_y = -mg)\)

When it reaches its peak height, all of the kinetic energy at \(t_3\) is converted back into potential energy.

If the collision were truly elastic, the ball will return to its original height \(h_2\).

If the collision is inelastic, some energy is converted into heat and \(h_y < h_2\).
Question: How can momentum be conserved when the ball hits the earth!?  
Seems like $\vec{p}_{\text{total}}$ completely changes direction!

\[ \begin{align*} \downarrow & \vec{p}_z \\ \downarrow & \vec{p}_z \end{align*} \]

What's really going on is that the ball is falling down and the earth (the entire planet!) is falling up!

Newton's 3rd law: \[ \vec{F}_{\text{ball on E}} = -\vec{F}_{\text{E on ball}} \]
\[ F_{\text{E on ball}} = mg = F_{\text{ball on E}} \]

For a 1 kg ball, $F = 9.8$ N

The acceleration of the earth is

\[ a_e = \frac{F}{M} = \frac{9.8 \text{ N}}{5.97 \times 10^{24} \text{ kg}} = 1.6 \times 10^{-24} \text{ m/s}^2 \]

(tiny!)
If you drop the 1 kg ball from 3 m, right before the ball hits the ground, the earth has moved a tiny bit up toward the ball.

By how much?

\[ y_E = \frac{1}{2} a_E t^2 \]

Time for ball to fall

\[ h = \frac{1}{2} g t^2 \Rightarrow t^2 = \frac{2h}{g} \]

\[ y_E = \frac{1}{2} \left( 1.6 \times 10^{-24} \text{ m/s}^2 \right) \left( \frac{2 \cdot 3 \text{ m}}{9.8 \text{ m/s}^2} \right) \]

\[ y_E = 4.9 \times 10^{-25} \text{ m}! \]

Size of an atom is ~10^{-10} m

...a nucleus is ~10^{-15} m

This is ten billion times smaller than a nucleus!

Momentum is conserved. It just doesn't look that way...
Compare: two balls of equal mass and equal and opposite momenta collide head-on

Before: \( t_1 \)

\[ \bigcirc \rightarrow \vec{p} \leftarrow \vec{p} \]

Total \( \vec{p} \) is zero

After: \( t_2 \)

\[ \vec{p} = ma \]

Now, a very heavy ball and a very light ball with equal and opposite momenta

\[ \bigcirc \rightarrow \vec{p} \leftarrow \vec{p} \]

\[ p_L = m v_L \quad \text{and} \quad p_H = M v_H \]

If \( p_L = p_H \), \( m v_L = M v_H \)

\[ v_H = v_L \frac{m}{M} \]

\( v_H \) very small because \( M \) is very large!
Bouncing basket ball
and tennis ball

drop a tennis ball resting on a
basket ball and see what
happens!

\[ h \]

\[ t_1 \]

just before
bounce

\[ t_2 \]

just after
Basket ball
bounce

\[ t_3 \]

\[ \vec{P}_B \]

\[ \vec{P}_B \]

\[ \vec{P}_T \]

\[ \vec{P}_T \]

\[ t_4 \]

just after
Basket ball
Hits
tennis ball

\[ P_{t4} > P_{t3} \]
Can you qualitatively explain why the tennis ball goes much higher than its original height using:
- conservation of momentum
- "" energy?

Another example:
5 steel balls hanging in a frame.

When they are in this position, the net force on each ball is zero (it's not accelerating).

The collisions between the steel balls are elastic: total momentum and total kinetic energy are conserved.

First, consider just two balls...
Let one swing down and hit the second one, which is at rest.

\[ \text{just before collision} \]

1. **Momentum conservation**:
   \[ MV_A = MV_A' + MV_B' \]

2. **Kinetic energy conservation**:
   \[ \frac{1}{2} MV_A^2 = \frac{1}{2} MV_A'^2 + \frac{1}{2} MV_B'^2 \]

1. \[ V_A = V_A' + V_B' \]

2. \[ V_A^2 = V_A'^2 + V_B'^2 \]

\[ (V_A' + V_B')^2 = V_A'^2 + V_B'^2 \]
\[ V_A'^2 + V_B'^2 + 2V_A'V_B' = V_A'^2 + V_B'^2 \]
\[ 2V_A'V_B' = 0 \]
\[ V_A' \text{ must be zero} \Rightarrow V_B' = V_A \]
When you swing 3 balls, 3 come up on the other side.

or 2 and 2
or 1 and 1

before

after
Pendulum with two unequal masses

$t_1$: let go of $m$

$t_2$: just before first collision

$t_3$: just after collision

$P_{m1} - P_{m3} = P_{m2}$

$h_4 < h_1$
$t_5$: just before 2nd collision

$t_6$: just after 2nd collision

$P_{m6} = -P_{m2}$

Can you describe this using conservation of energy and momentum?

$t_7$: light mass returns to original height.